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A MODEL FOR γ - DECAY OF ATOMIC OR NUCLEAR SYSTEMS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled A MODEL FOR γ - DECAY OF ATOMIC OR NUCLEAR SYSTEMS, submitted by Edgar Alton Henley, Jr. in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

In this thesis we consider the problem of a charged, bound particle interacting with an electromagnetic field.

First, we consider a classical model for the interaction of a bound, charged particle with an electromagnetic field. We determine the frequency shift of the oscillating particle due to the electromagnetic field.

Second, we consider a quantum mechanical model for a spinless bound particle interacting with an electromagnetic field. We determine the level shift, line breadth, and the probability of transition for this model.

Finally, we apply the results obtained from the quantum mechanical model to the electromagnetic transition of an atomic and a nuclear system. From this application we form some general conclusions as to the dependence of line breadth and level shift on the binding potential.

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CHAPTER I. INTRODUCTION

If a system which consists of a bound, charged particle is placed in an electromagnetic field there will be an interaction between the particle and the field. Due to this interaction, the particle can do one of two things; first, the particle may absorb a photon and be excited to a higher state or, second, if the particle is in an excited state it may be induced to emit a photon and make a transition to a lower state. Due to the interaction with the electromagnetic field, the energy level of the particle is shifted to a new level, also when a transition occurs the line width is not a sharp line but has a finite width due to the radiative damping effect.[1]

The problem of radiative transitions has been studied by many different people since 1916 when Einstein[2] presented his derivation of his now famous coefficients of absorption, spontaneous emission and induced emission. It was not until 1926, however, that Dirac[3] really got on the right track in tackling this problem. In his work, Dirac presented the first satisfactory theory of radiation, in which he derived the Einstein formulas.

The work on the quantum theory of radiation has been advanced by many people and this work has resulted in the treatises of Fermi[4] and Heitler,[5] wherein reference to the earlier work can be found.

The common approach to the problem of radiative transitions is through the use of perturbation theory.[6] In perturbation theory the interaction term is treated as a small perturbation on the system

of the electromagnetic field plus the bound particle. While the perturbation approach yields relatively good results, it would be better if the problem could be solved exactly. Unfortunately, this is an unrealized dream for Physicists at present. There are several reasons why an exact solution is preferred over a perturbative one. First, there is the aesthetic pleasure of being able to solve the problem exactly. Second, and most important, is that to use perturbation theory implies an assumption of analyticity in the coupling constant. This may or may not be the case; it is as yet unresolved. Third, there are cases where the coupling constants are quite large, i.e. nuclear forces, and a correct perturbation treatment is impossible. Since the complete Hamiltonian for a charged particle in an electromagnetic field cannot be solved exactly, we shall not use the complete Hamiltonian in this work, but a model which can be solved exactly.

In the second chapter we consider a classical model of a bound, charged particle interacting with an electromagnetic field. The particle is assumed to be bound in a harmonic oscillator potential and oscillates with a frequency, ν. The shift in this frequency is calculated and is shown to be of the order e^2, where e is the charge of an electron.

In the third chapter we consider a non-relativistic quantum mechanical model of a spinless bound particle interacting with an electromagnetic field. The particle is assumed to be bound in a central potential, which may be the coulomb field of the nucleus in the case of an electron; or nuclear field in the case of a nucleon. We calculate the line breadth, level shift and the transition probability. The line

breadth and level shift are shown to be of order e^2, where e is the charge of an electron. The results we obtain agree with the results of Chan and Razavy[7] in their work on the theory of natural line breadth.

In the fourth chapter, we apply the results of the third chapter to an atomic and a nuclear system. We give a general discussion on the dependence of the line breadth and level shift on the form of the binding potential. We find that in general a tightly-bound particle has its energy level shifted by a very small amount and its natural line width is quite small, while for a loosely-bound particle we find that the level shift and natural line width is quite large.

CHAPTER II. THE CLASSICAL MODEL

In this chapter we shall discuss the classical problem of a charged particle, bound by a harmonic oscillator potential, interacting with an electromagnetic field. The Hamiltonian we shall use is[8]

$$H = (1/2m)[\vec{P}(t)-e\int\vec{A}(\vec{r},t)\rho(\vec{r})d^3r]^2 + (1/2)m\nu^2\vec{R}^2(t)$$

$$+ (1/8\pi)\int\{[\nabla\times\vec{A}(\vec{r},t)]^2 + 16\pi^2\vec{\Phi}^2(\vec{r},t)\}d^3r \qquad (2\text{-}1)$$

where $\rho(\vec{r})$ is the charge distribution function which we take as

$$\rho(\vec{r}) = (\beta^2/4\pi r)e^{-\beta r} \qquad (2\text{-}2)$$

where

$$\beta = 1/r_0 \qquad (2\text{-}3)$$

and where r_0 is the mean square radius of the charge distribution. r_0 is assumed to be very small, approaching but never reaching zero, i.e. β is very large. In 2-1, $\vec{A}(\vec{r},t)$ is the vector potential, $\vec{\Phi}(\vec{r},t)$ is the momentum conjugate to $\vec{A}(\vec{r},t)$, m is the mass of the particle, ν is the frequency of oscillation of the particle, $\vec{R}(t)$ is the position of the particle, and $\vec{P}(t)$ is the momentum conjugate to $\vec{R}(t)$.

The first term in 2-1 is the mechanical momentum of the system which contains both the energy of the particle plus the energy of interaction. The second term is the potential by which the particle is bound. The third term is the energy of the electromagnetic field.

The equations of motion obtained from 2-1 are

$$\partial_t\vec{A}(\vec{r},t) = \delta H/\delta\vec{\Phi} = 4\pi\vec{\Phi}(\vec{r},t) \tag{2-4}$$

$$\partial_t\vec{\Phi}(\vec{r},t) = -\delta H/\delta\vec{A} = -(1/4\pi)\ \mathrm{curl\ curl}\ \vec{A}(\vec{r},t) + (e/m)\rho(\vec{r})\ [\vec{P}(t)-e\int\vec{A}(\vec{r},t)\rho(\vec{r})d^3r] \tag{2-5}$$

$$\partial_t\vec{P}(t) = -\delta H/\delta\vec{R} = -m\nu^2\vec{R}(t) \tag{2-6}$$

$$\partial_t\vec{R}(t) = \delta H/\delta\vec{P} = (1/m)[\vec{P}(t)-e\int\vec{A}(\vec{r},t)\rho(\vec{r})d^3r]. \tag{2-7}$$

From 2-4 we obtain

$$\partial^2{}_t\vec{A}(\vec{r},t) = 4\pi\ \partial_t\vec{\Phi}(\vec{r},t). \tag{2-8}$$

Substituting 2-5 into 2-8 we obtain

$$\partial^2{}_t\vec{A}(\vec{r},t) = -\mathrm{curl\ curl}\ \vec{A}(\vec{r},t) + (4\pi e/m)\rho(\vec{r})[\vec{P}(t)-e\int\vec{A}(\vec{r},t)\rho(\vec{r})d^3r]. \tag{2-9}$$

Using the coulomb gauge, where div $\vec{A}(\vec{r},t) = 0$, we then have

$$\partial^2{}_t\vec{A}(\vec{r},t) = \nabla^2\vec{A}(\vec{r},t) + (4\pi e/m)\rho(\vec{r})[\vec{P}(t)-e\int\vec{A}(\vec{r},t)\rho(\vec{r})d^3r]. \tag{2-10}$$

From 2-7 we obtain

$$\partial^2{}_t\vec{R}(t) = (1/m)\partial_t\vec{P}(t)-(e/m)\int\partial_t\vec{A}(\vec{r},t)\rho(\vec{r})d^3r. \tag{2-11}$$

Substituting 2-6 into 2-11 we then have

$$\partial^2{}_t\vec{R}(t) = -\nu^2\vec{R}(t)-(e/m)\int\partial_t\vec{A}(\vec{r},t)\rho(\vec{r})d^3r. \tag{2-12}$$

Equations 2-10 and 2-12 are the equations we now wish to consider. Let

$$\vec{R}(t) = \sum_\omega \vec{R}_\omega e^{i\omega t} \tag{2-13}$$

$$\vec{A}(\vec{r},t) = \sum_{\omega}\vec{A}_{\omega}(\vec{r})e^{i\omega t}. \tag{2-14}$$

Substituting 2-13 and 2-14 into 2-10 and 2-12 we obtain

$$(\omega^2-\nu^2)\vec{R}_{\omega} = (i\omega e/m)\int\vec{A}_{\omega}(\vec{r})\rho(\vec{r})d^3r \tag{2-15}$$

$$(\nabla^2+\omega^2)\vec{A}_{\omega}(\vec{r}) = -4\pi ei\omega\rho(\vec{r})\vec{R}_{\omega}. \tag{2-16}$$

Since $\rho(\vec{r})$, 2-2, is zero except very close to the origin we need to substitute in place of $\vec{A}_{\omega}(\vec{r})$ the average value of $\vec{A}_{\omega}(\vec{r})$ as $r\to 0$. We calculate the average value of $\vec{A}_{\omega}(\vec{r})$ as follows. Let $\vec{A}_{\omega}(\vec{r})$ be given by

$$\vec{A}_{\omega}(\vec{r}) = \sum_{\lambda}\vec{e}_{\lambda}(\omega)B_{\omega}^{\lambda}\{\sin\omega(L-r)\}/r \tag{2-17}$$

where the sum is over the directions of polarization, and $\omega = |\vec{k}|$, where $\vec{k}$ is the momentum of the field. We can write $\vec{A}_{\omega}(\vec{r})$ as

$$\vec{A}_{\omega}(\vec{r}) = \vec{A}_{\omega}^{\ell}(\vec{r}) + \vec{A}_{\omega}^{\tau}(\vec{r}) \tag{2-18}$$

where $\vec{A}_{\omega}^{\ell}(\vec{r})$ is che longitudinal part and $\vec{A}_{\omega}^{\tau}(\vec{r})$ the transverse part of $\vec{A}_{\omega}(\vec{r})$, where curl $\vec{A}_{\omega}^{\ell}(\vec{r})$ = div $\vec{A}_{\omega}^{t}(\vec{r}) = 0$. The singularity of $\vec{A}_{\omega}(\vec{r})$ near the origin is $1/r$ therefore near the origin we can write $\vec{A}_{\omega}(\vec{r}) = \vec{e}_{\lambda}(\omega)/r$; then we require div $\{\vec{e}_{\lambda}(\omega)/r\}$ = div $\vec{A}_{\omega}^{\ell}(\vec{r})$. Since curl $\vec{A}_{\omega}^{\ell}(\vec{r}) = 0$ we can write $\vec{A}_{\omega}^{\ell}(\vec{r}) = \nabla f(\vec{r})$; therefore

$$\text{div}\ \{\vec{e}_{\lambda}(\omega)/r\} = \nabla^2 f(\vec{r})$$

or

$$f(\vec{r}) = (1/2)\vec{e}_{\lambda}(\omega)\cdot\nabla r. \tag{2-19}$$

From 2-19 we have

$$\vec{A}^{\ell}_{\omega}(\vec{r}) = (1/2)\nabla\{\vec{e}_{\lambda}(\omega)\cdot\nabla\}r. \tag{2-20}$$

From 2-18 and 2-20 we obtain

$$\begin{aligned}\vec{A}^{\tau}_{\omega}(\vec{r}) &= \vec{A}_{\omega}(\vec{r})-\vec{A}^{\ell}_{\omega}(\vec{r}) = \{\vec{e}_{\lambda}(\omega)/r\}-(1/2)\nabla\{\vec{e}_{\lambda}(\omega)\cdot\nabla\}r \\ &= \{\vec{e}_{\lambda}(\omega)/2r\} + \{[\vec{e}_{\lambda}(\omega)\cdot\vec{r}]/2r^3\}\vec{r}.\end{aligned}$$

Consider

$$\vec{e}_{\lambda}(\omega)\cdot\vec{A}^{\tau}_{\omega}(\vec{r}) = (1/2r)[1 + \{\vec{e}_{\lambda}(\omega)\cdot\vec{r}\}^2/r^2] = (1/2r)[1 + \cos^2\theta] \tag{2-21}$$

where $\cos\theta = \vec{e}_{\lambda}(\omega)\cdot\vec{r}/r$. Using 2-21, the average value of $\vec{e}_{\lambda}(\omega)\cdot\vec{A}^{\tau}_{\omega}(\vec{r})$ is then

$$\overline{\vec{e}_{\lambda}(\omega)\cdot\vec{A}^{\tau}_{\omega}(\vec{r})} = (1/2r)\int_0^{\pi}[1 + \cos^2\theta]\sin\theta d\theta/\int_0^{\pi}\sin\theta d\theta = 2/3r. \tag{2-22}$$

We therefore write

$$\overline{\vec{A}_{\omega}(\vec{r})} = \sum_{\lambda}(2B^{\lambda}_{\omega}/3r)\vec{e}_{\lambda}(\omega)\sin\omega(L-r). \tag{2-23}$$

Substituting 2-23 into 2-15 we obtain

$$(\omega^2-\nu^2)\vec{R}_{\omega} = (2i\omega e/3m)\sum_{\lambda}B^{\lambda}_{\omega}\vec{e}_{\lambda}(\omega)\int\rho(\vec{r})\{\sin\omega(L-r)/r\}d^3r \tag{2-24}$$

or

$$\vec{R}_{\omega} = (2i\omega e/3m)(1/\omega^2-\nu^2)\sum_{\lambda}B^{\lambda}_{\omega}\vec{e}_{\lambda}(\omega)\int\rho(\vec{r})\{\sin\omega(L-r)/r\}d^3r. \tag{2-25}$$

Substituting 2-25 into 2-16 we obtain

$$(\nabla^2+\omega^2)\vec{A}_{\omega}(\vec{r}) = \{8\pi\omega^2e^2\rho(\vec{r})/3m^2(\omega^2-\nu^2)\}\sum_{\lambda}B^{\lambda}_{\omega}\vec{e}_{\lambda}(\omega)\int\rho(\vec{r})\{\sin\omega(L-r)/r\}d^3r. \tag{2-26}$$

Substituting 2-17 into 2-26 we obtain

$$(\nabla^2+\omega^2)\{\sin\omega(L-r)/r\} = \{8\pi\omega^2e^2\rho(\vec{r})/3m^2(\omega^2-\nu^2)\}\int\rho(\vec{r})\{\sin\omega(L-r)/r\}d^3r. \tag{2-27}$$

Substituting

$$\nabla^2\{\sin\omega(L-r)/r\} = -\omega^2\{\sin\omega(L-r)/r\}+\sin\omega(L-r)\nabla^2(1/r) \tag{2-28}$$

into 2-27 we obtain

$$\sin\omega(L-r)\nabla^2(1/r) = \{8\pi e^2\omega^2/[3m(\omega^2-\nu^2)]\}\rho(\vec{r})\int\rho(\vec{r})\{\sin\omega(L-r)/r\}d^3r. \tag{2-29}$$

The integral occuring in 2-29 can be written as

$$\begin{aligned}\int\rho(\vec{r})\{\sin\omega(L-r)/r\}d^3r &= \beta^2\int_0^\infty e^{-\beta r}\sin\omega(L-r)dr \\ &= \beta^2\int_0^\infty e^{-\beta r}[\sin\omega L\cos\omega r-\cos\omega L\sin\omega r]dr \\ &= \sin\omega L\{\beta/[1+(\omega^2/\beta^2)]\} - \omega\cos\omega L\{1/[1+(\omega^2/\beta^2)]\end{aligned} \tag{2-30}$$

where we have used 2-2.

With the assumption that β is very large we can neglect the second term in the denominator of each of the terms in 2-30, we then obtain

$$\int\rho(\vec{r})\{\sin\omega(L-r)/r\}d^3r = \{\sin\omega L/r_0\} - \omega\cos\omega L \tag{2-31}$$

where we have used 2-3. Since $\nabla^2(1/r)$ is given by[9]

$$\nabla^2(1/r) = -4\pi\delta(\vec{r}) \tag{2-32}$$

we can write 2-29 as

$$-\sin\omega(L-r)\delta(\vec{r}) = \{2e^2\omega^2/[3m(\omega^2-\nu^2)]\}[(\sin\omega L/r_0)-\omega\cos\omega L]\rho(\vec{r}) \tag{2-33}$$

where we have used 2-31. Integrating 2-33 over all space we obtain

$$-\sin\omega L = \{2e^2\omega^2/[3m(\omega^2-\nu^2)]\}[(\sin\omega L/r_0) - \omega\cos\omega L] \tag{2-34}$$

where we have used the fact

$$(\beta^2/4\pi)\int(e^{-\beta r}/r)d^3r = 1. \tag{2-35}$$

From 2-34 we obtain the eigenvalue equation for ω, i.e.

$$\tan\omega L-2e^2\omega^3/\{3m(\omega^2-\nu^2)+2e^2\omega^2/r_0\} = F(\omega) = 0. \tag{2-36}$$

We note that if $e\to0$ in 2-36 we obtain the eigenvalue equation for the uncoupled system, i.e.

$$\nu_n = n\pi/L;\ n = 0,1,2,... \tag{2-37}$$

We now want to show that for each ν there is one and only one ω. Consider 2-36, where $n\pi\leq\omega L\leq(n+1)\pi$. There are two cases to consider in studying this equation. First there is the case where n is such that

$$3m(\omega_n^2-\nu^2) + (2e^2\omega_n^2/r_0) = 0, \tag{2-38}$$

i.e. the resonance condition. Second we have to study the case for all other values of n. First we consider the second case. If we draw a sketch of $F(\omega)$ versus ω, where

$$F(\omega) = \tan\omega L-2e^2\omega^3/\{3m(\omega^2-\nu^2)+2e^2\omega^2/r_0\}$$

For $n\pi \leq \omega L \leq (n+1)\pi$ we obtain

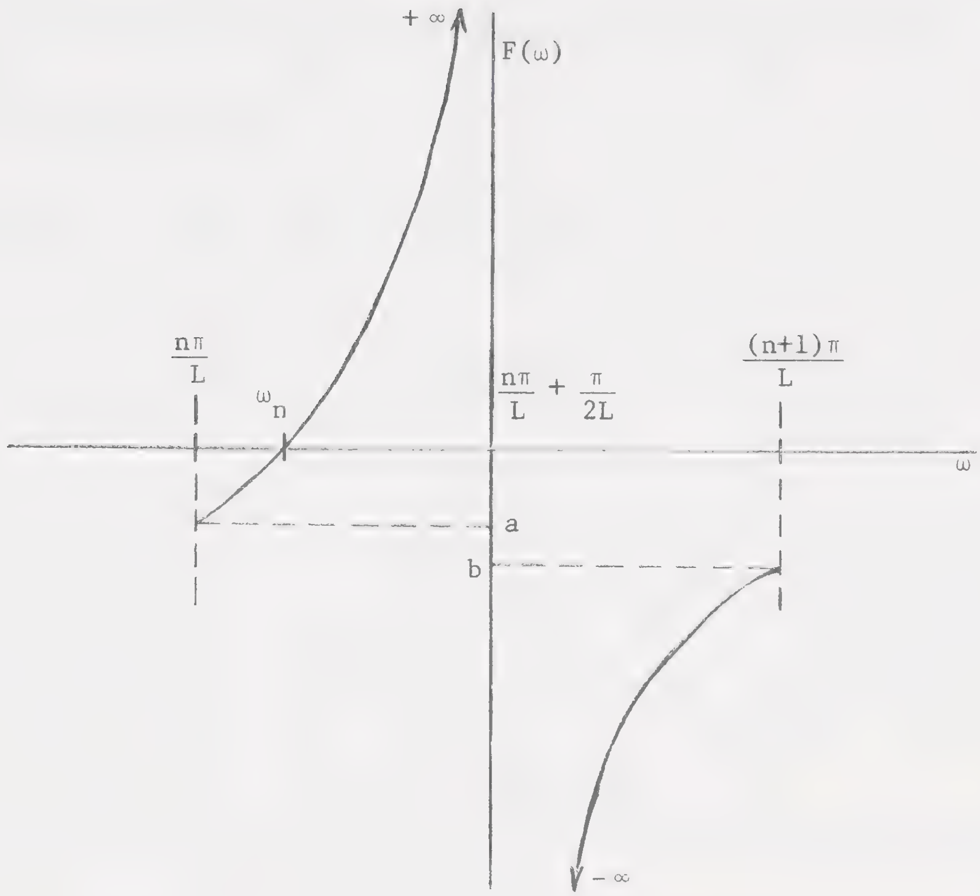

Figure 1.

where

$$a = -\frac{2e^2[n\pi/L]^3}{3m[(n\pi/L)^2-\nu^2] + (2e^2/r_0)[n\pi/L]^2}$$

$$b = -\frac{2e^2[(n+1)\pi/L]^3}{3m\{[(n+1)\pi/L]^2-\nu^2\} + (2e^2/r_0)[(n+1)\pi/L]^2} .$$

We see from figure 1 that 2-36 only has one root in each interval $n\pi \leq \omega L \leq (n+1)\pi$. We also note that the root of 2-36 satisfies $n\pi \leq \omega_n L \leq n\pi + \pi/2$. We also notice that as $n \to \infty$ the value of $\omega_n \to n\pi/L + \pi/2L$.

We see from 2-38 that for resonance ω_R satisfies

$$\omega_R = \nu/[1 + (2e^2/3mr_0)]^{1/2}.$$

For this case we obtain the following figure

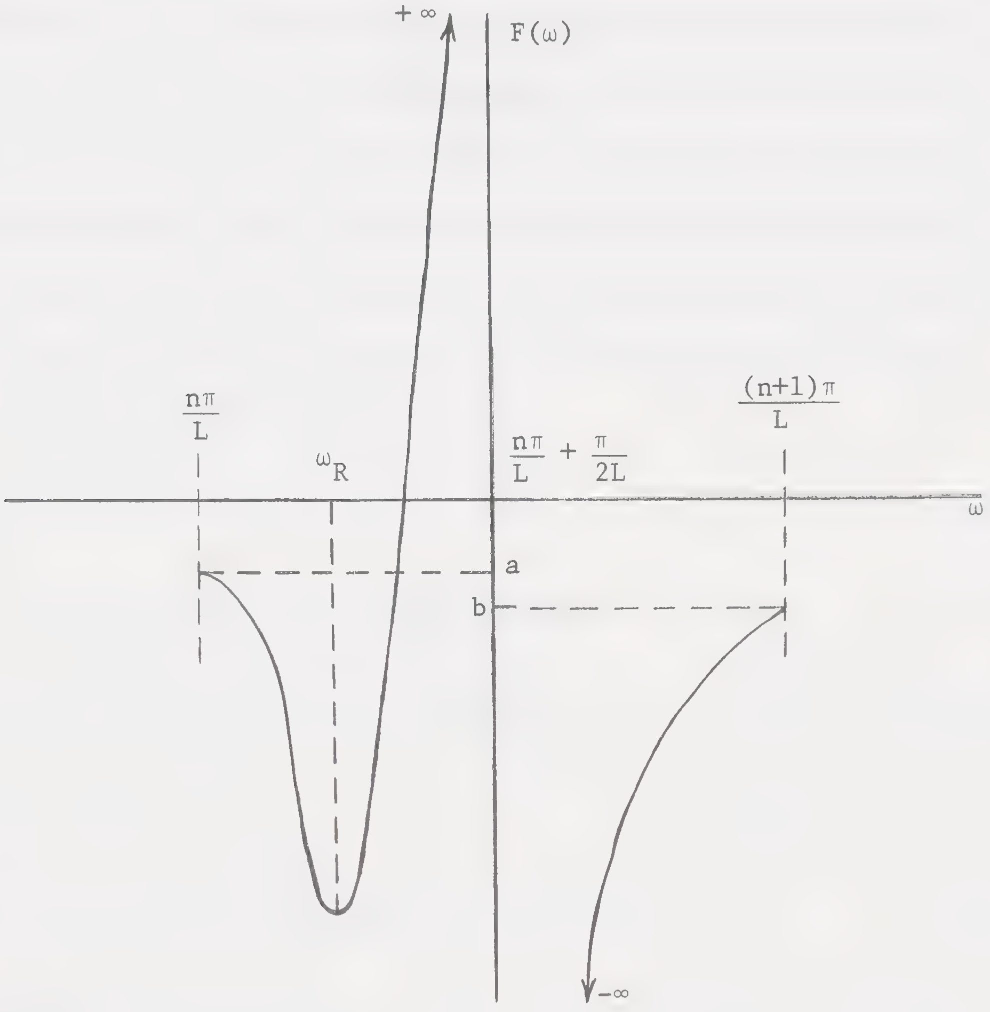

Figure 2.

From 2-36 for each ω_n such that 2-38 is not satisfied we obtain

$$\tan\omega_n L = 2e^2\omega_n^3 / [3m(\omega_n^2-\nu^2) + 2e^2\omega_n^2/r_0]. \qquad (2\text{-}39)$$

It is important to note that ν in 2-39 is not one of the ν_n in 2-37. ν is the frequency of oscillation of the particle due to the harmonic force in the absence of the electromagnetic field. The ν_ns are the frequencies of the electromagnetic field in the absence of coupling to the particle system. The ω_ns are the new frequencies of the coupled system, ω_0 is the shifted frequency of the oscillating particle and the other ω_ns are the shifted frequencies of the electromagnetic field.

From 2-37 we have

$$\tan\nu_n L = 0. \qquad (2\text{-}40)$$

From 2-39 and 2-40 we then obtain

$$\tan(\omega_n-\nu_n)L = 2e^2\omega_n^3[3m(\omega_n^2-\nu^2) + 2e^2\omega_n^2/r_0] \qquad (2\text{-}41)$$

or

$$\omega_n-\nu_n = (1/L)\mathrm{Arctan}\{2e^2\omega_n^3/[3m(\omega_n^2-\nu^2) + 2e^2\omega_n^2/r_0]\}. \qquad (2\text{-}42)$$

Since the argument of Arctan, in 2-47, is very small we can write

$$\omega_n-\nu_n = (1/L)\{2e^2\omega_n^3/[3m(\omega_n^2-\nu^2) + 2e^2\omega_n^2/r_0]\}. \qquad (2\text{-}43)$$

For n = 0 we obtain from 2-43

$$\omega_0 = (1/L)\{2e^2\omega_0^3/[3m(\omega_0^2-\nu^2) + 2e^2\omega_0^2/r_0]\}. \qquad (2\text{-}44)$$

Since we are mainly interested in the shift in the oscillating particle frequency we will not consider the equation for $n = 1,2,3\ldots$.

From 2-44 we can write

$$\omega_0^2 = \nu^2 - [2e^2\omega_0^2/(3mr_0)] + [2e^2\omega_0^2/(3mL)] \tag{2-45}$$

or putting $\omega_0 = \nu$, since $\omega_0-\nu \sim e^2$, on the right hand side we have

$$\omega_0 = \nu\{1-[2e^2/(3mr_0)] + [2e^2/(3mL)]\}^{1/2}. \tag{2-46}$$

Since r_0 is assumed to be very small we can neglect the last term in the parentheses in 2-46, we then have

$$\omega_0 = \nu[1-2e^2/(3mr_0)]^{1/2} \tag{2-47}$$

or if $r_0 > 2e^2/3m$ we can write

$$\omega_0-\nu = -[e^2/(3mr_0)]\nu. \tag{2-48}$$

We see that the frequency shift is of order e^2 and proportional to ν.

We notice that if r_0 is decreased to zero in 2-47 we obtain imaginary values for ω_0. These imaginary values correspond to the phenomenon of self-acceleration.[10] We see that the minimum value that r_0 can assume such that ω_0 is real is

$$r_0 = 2e^2/3m. \tag{2-49}$$

The classical radius of the electron is e^2/m, thus we see that r_0 is approximately equal to the classical radius of the electron.

The phenomenon of self-acceleration arises because there are not enough initial conditions placed on the problem.[11] One assumption,

first considered by Dirac,[12] which eliminates these unphysical solutions is that the acceleration goes to zero as time approaches infinity.

It should also be noted that no classical theory can explain the structure of the electron and therefore all structural concepts, such as radius, should not occur in the classical theory. The structural properties should be explained, if indeed they can be explained at all, within the framework of quantum theory.[13]

CHAPTER III. THE QUANTUM MECHANICAL MODEL

In this chapter we consider a quantum mechanical model for the non-relativistic interaction between a bound, spinless, charged particle and the electromagnetic field. An example of such a system is the nucleons in the nucleus interacting with an electromagnetic field. We shall consider only processes involving a single charged particle with the emission or absorption of one photon.

In section 1 we discuss the model and obtain the Hamiltonian we shall be using. In section 2 we derive the equations of motion which we solve in section 3 and obtain expressions for the level shift, line breadth, and the cross-section for emission or absorption of a photon.

1. The Hamiltonian

The complete Hamiltonian for a non-relativistic, bound, spinless, charged particle interacting with an electromagnetic field is,[14] ($\hbar = c = 1$)

$$H = H_0 + H_1 \tag{3-1}$$

where

$$\begin{aligned} H_0 = {} & \int\{(1/2m)\nabla\psi^*(\vec{r},t)\cdot\nabla\psi(\vec{r},t) + V(\vec{r})\psi^*(\vec{r},t)\psi(\vec{r},t)\}d^3r \\ & +(1/8\pi)\int\{\vec{E}^2(\vec{r},t) + [\nabla\times\vec{A}(\vec{r},t)]^2\}d^3r \end{aligned} \tag{3-2}$$

and

$$H_1 = (ie/2m)\int\vec{A}(\vec{r},t)\cdot\{\psi^*(\vec{r},t)\nabla\psi(\vec{r},t)-\nabla\psi^*(\vec{r},t)\psi(\vec{r},t)\}d^3r$$
$$+ (e^2/2m)\int\vec{A}^2(\vec{r},t)\psi^*(\vec{r},t)\psi(\vec{r},t)d^3r$$
$$+ (e^2/2)\int\int\psi^*(\vec{r},t)\psi^*(\vec{r}',t)\psi(\vec{r}',t)\psi(\vec{r},t)/|\vec{r}-\vec{r}'|\, d^3r d^3r'. \qquad (3\text{-}3)$$

The first term in 3-2 describes the charged particle, mass m, bound in a potential $V(\vec{r})$. The second term in 3-2, where we use Gaussian units, is the radiation energy of the electromagnetic field. Both $\vec{E}(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ are transverse fields.

The first and second term in 3-3 describe the interaction between the charged particle and the electromagnetic field. The third term in 3-3 is the electrostatic self-energy and can be absorbed into $V(\vec{r})$ in 3-2.

In the model we shall study we modify the first and second terms of 3-3 so as to obtain a Hamiltonian which can be solved exactly. The justification for these modifications is mainly that the resulting problem can be solved exactly. Another justification is that this model explains some of the experimentally observed properties of the interaction of bound charged particles and an electromagnetic field. Also, once we have obtained an exact solution of our model it would be possible, not done here, to apply perturbation theory and thereby solve the complete problem, where the perturbation used is the difference between our model and the exact Hamiltonian.

The Hamiltonian in 3-1 is the classical Hamiltonian, therefore it must be symmetrized before it can be used in the Quantum Mechanical formulation of the problem. This symmetrization is needed since the Hamiltonian, or any observable, is required to be Hermitian. We will therefore have to symmetrize the Hamiltonian we use for our model.

The Hamiltonian we shall consider is

$$H = H_0 + H' \tag{3-4}$$

where

$$H_0 = \sum_{\alpha=i,f} \int \{(1/2m)\nabla\psi^*_\alpha(\vec{r},t)\cdot\nabla\psi_\alpha(\vec{r},t) + V(\vec{r})\psi^*_\alpha(\vec{r},t)\psi_\alpha(\vec{r},t)\}d^3r$$

$$+ (1/8\pi)\int\{\vec{E}^2(\vec{r},t) + [\nabla\times\vec{A}(\vec{r},t)]^2\}d^3r \tag{3-5}$$

and

$$H' = (ie/2m)\int\vec{A}^+(\vec{r},t)\cdot\{\psi^*_i(\vec{r},t)\nabla\psi_f(\vec{r},t)-\nabla\psi^*_i(\vec{r},t)\psi_f(\vec{r},t)\}d^3r$$

$$+ (ie/2m)\int\vec{A}^-(\vec{r},t)\cdot\{\psi^*_f(\vec{r},t)\nabla\psi_i(\vec{r},t)-\nabla\psi^*_f(\vec{r},t)\psi_i(\vec{r},t)\}d^3r$$

$$+ (e^2/2m)\sum_{\alpha=i,f}\int\psi^*_\alpha(\vec{r},t)\{\vec{A}^+(\vec{r},t)\cdot\vec{A}^-(\vec{r},t)+\vec{A}^-(\vec{r},t)\cdot\vec{A}^+(\vec{r},t)\}\psi_\alpha(\vec{r},t)d^3r \tag{3-6}$$

where we have symmetrized H so that it is Hermitian. In 3-5 and 3-6, i and f refer to the initial and final states of the system. $\vec{A}^+(\vec{r},t)$ and $\vec{A}^-(\vec{r},t)$ are the positive and negative frequency parts of the electromagnetic field. Assuming periodic boundary conditions and unit normalization volume we can write,[15]

$$\vec{A}^{\pm}(\vec{r},t) = \sum_{k,\lambda}(2\pi/k)^{1/2}\vec{e}_\lambda(k)\left\{\begin{matrix}a_{k,\lambda}(t)\\ a^\dagger_{k,\lambda}(t)\end{matrix}\right\}\exp\{\pm i\vec{k}\cdot\vec{r}\} \tag{3-7}$$

where the photon energy is $k = |\vec{k}|$ and $\vec{e}_\lambda(k)$ is the unit polarization vector orthogonal to $\vec{k}$. The sum over k extends over all plane wave momentum states $\vec{k}$ in the unit normalized volume. The sum over λ extends over the two allowed directions of polarization of a photon having a momentum $\vec{k}$. $a^\dagger_{k,\lambda}(t)$ $[a_{k,\lambda}(t)]$ is the creation [annihilation] operator for a photon of momentum $\vec{k}$ and polarization $\vec{e}_\lambda(k)$. We also have,[16]

$$\vec{A}(\vec{r},t) = \vec{A}^{+}(\vec{r},t) + \vec{A}^{-}(\vec{r},t)$$

$$\vec{E}(\vec{r},t) = -\,\partial_t \vec{A}(\vec{r},t). \tag{3-8}$$

Defining $U_\alpha(\vec{r})$ as the eigenfunction of the Schrödinger equation

$$\{(-1/2m)\nabla^2 + V(\vec{r})\}U_\alpha(\vec{r}) = E_\alpha U_\alpha(\vec{r}) \qquad \alpha = i,f \tag{3-9}$$

we then can expand $\psi_\alpha(\vec{r},t)$ and $\psi^*_\alpha(\vec{r},t)$ as

$$\psi_\alpha(\vec{r},t) = b_\alpha(t)U_\alpha(\vec{r})$$

$$\psi^*_\alpha(\vec{r},t) = b^\dagger_\alpha(t)U^*_\alpha(\vec{r}) \tag{3-10}$$

where $b^\dagger_\alpha(t)$ $[b_\alpha(t)]$ is the creation [annihilation] operator for a charged particle in the eigenstate α.

The $b_\alpha(t)$ and $a_{k,\lambda}(t)$ satisfy the following commutation relations

$$[b_\alpha(t), b^\dagger_{\alpha'}(t)]_+ = \delta_{\alpha,\alpha'}$$

$$[a_{k,\lambda}(t), a^\dagger_{k',\lambda'}(t)] = \delta_{k,k'}\delta_{\lambda,\lambda'} \tag{3-11}$$

with all other pairs commuting.

Substituting 3-7 and 3-10 into 3-4 we obtain the following operator form of our Hamiltonian

$$H = E_f b_f^\dagger(t) b_f(t) + E_i b_i^\dagger(t) b_i(t) + \sum_k k[a_k^\dagger(t) a_k(t) + (1/2)]$$
$$+ e\sum_k (1/2k)^{1/2} [\beta(k) b_i^\dagger(t) b_f(t) a_k(t) + \beta^*(k) a_k^\dagger(t) b_f^\dagger(t) b_i(t)]$$
$$+ (\pi e^2/2m) \sum_{q,k} \sum_{\alpha=i,f} (1/kq)^{1/2} [b_\alpha^\dagger(t) b_\alpha(t) B_\alpha(k-q)][2a_q^\dagger(t) a_k(t) + \delta_{qk}] \tag{3-12}$$

where the sum over λ is suppressed, and where

$$\beta(k) = (i/m)(4\pi)^{1/2} \int \exp(i\vec{k}\cdot\vec{r}) U_i(\vec{r}) \vec{e}(k)\cdot\nabla U_f(\vec{r}) d^3r \tag{3-13}$$

and

$$B_\alpha(k-q) = \vec{e}(k)\cdot\vec{e}(q) \int \exp[i(\vec{k}-\vec{q})\cdot\vec{r}] U_\alpha^*(\vec{r}) U_\alpha(\vec{r}) d^3r. \tag{3-14}$$

If the last term in 3-12 is dropped we obtain the same Hamiltonian as studied by Chan and Razavy[17] in connection with the problem of natural line width.

The Hamiltonian 3-12 is in the Heisenberg representation; we can transform to the Schrödinger representation by the transformation of operators given by

$$O(t) = \exp(iHt) O(t=0) \exp(-iHt).$$

Thus in the Schrödinger representation the Hamiltonian has the same form but with all operators evaluated at $t = 0$. We shall drop the t parameter and assume all operators are evaluated at $t = 0$ and use the time-dependent Schrödinger equation to study the equations of motion in the next section.

2. The Equations of Motion

We shall assume that the initial condition of the system is given by

$$|\psi(0)\rangle = |0,1_i\rangle \tag{3-15}$$

i.e., the charged particle is in its initial state, with no photons present. Due to the form of the Hamiltonian, 3-12, $|\psi(t)\rangle$, the state after time t, must have the form[18]

$$|\psi(t)\rangle = C_0(t)|0,1_i\rangle + \sum_k C_k(t)|1_k,1_f\rangle \tag{3-16}$$

where $|1_k,1_f\rangle$ denotes the state with the charged particle in its final state and one photon of momentum $\vec{k}$ and polarization $\vec{e}_\lambda(k)$ present. Comparing 3-15 and 3-16 we see that at $t = 0$ we have

$$C_0(t = 0) = 1 \qquad\qquad C_k(t = 0) = 0. \tag{3-17}$$

Following the method of Haake and Weidlich[19] we write the Schrödinger equation

$$i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle \tag{3-18}$$

where H is given by 3-12 but with all operators evaluated at $t = 0$. Substituting 3-16 into 3-18 we obtain

$$i\partial_t C_0(t) = E_I C_0(t) + \sum_k [e\beta(k)/(2k)^{1/2}]C_k(t) \tag{3-19}$$

$$i\partial_t C_k(t) = (E_F + k)C_k(t) + [e\beta^*(k)/(2k)^{1/2}]C_0(t)$$
$$+ (\pi e^2/m)\sum_\ell [1/(k\ell)^{1/2}]B_f(\ell-k)C_\ell(t) \tag{3-20}$$

where

$$E_I = E_i + (\pi e^2/2m) \sum_k [B_i(0)/k] \tag{3-21}$$

and

$$E_F = E_f + (\pi e^2/2m)\sum_k [B_f(0)/k] + (1/2)\sum_k k. \tag{3-22}$$

The two divergent quantities, 3-21 and 3-22, are the renormalized energies of the initial and final states of the system. The expression 3-19 and 3-20 are the equations of motion of our system.

To find the normal modes of this coupled system of equations we write the Fourier series for $C_0(t)$ and $C_k(t)$ as[20]

$$C_n(t) = \sum_\omega p(\omega) C_n(\omega) \exp[-i(\omega + E_I)t] \qquad n = 0,k \tag{3-23}$$

where

$$p(\omega) = C_0^*(\omega) C_0(t = 0) + \sum_k C_k^*(\omega) C_k(t = 0). \tag{3-24}$$

Substituting 3-17 into 3-24 we have

$$p(\omega) = C_0^*(\omega). \tag{3-25}$$

Substituting 3-23 into 3-19 and 3-20 we obtain

$$\omega C_0(\omega) = e\sum_k [1/(2k)^{1/2}]\beta(k) C_k(\omega) \tag{3-26}$$

$$\omega C_k(\omega) = (E_F + k - E_I) C_k(\omega) + [e\beta^*(k)/(2k)^{1/2}] C_0(\omega)$$

$$+ (\pi e^2/m)\sum_\ell [1/(\ell k)^{1/2}] B_f(\ell - k) C_\ell(\omega). \tag{3-27}$$

By eliminating $C_0(\omega)$ between 3-26 and 3-27 we obtain an equation for $C_k(\omega)$

$$C_k(\omega) = -e^2\sum_{\ell} K(\ell,k)C_{\ell}(\omega) \tag{3-28}$$

where

$$K(\ell,k) = 1/[(E_F+k-E_I-\omega)(k\ell)^{1/2}]\{[\beta^*(k)\beta(\ell)]/2\omega+(\pi/m)B_f(\ell-k)\}. \tag{3-29}$$

In the next section we consider 3-28 and its solution and determine the level shift, line breadth, and cross-section.

3. Level Shift, Line Breadth and Probability of Decay

We can write 3-28 as

$$C(k,\omega) = -e^2\int_0^{\infty} C(\ell,\omega)K(\ell,k;\omega)d\ell \tag{3-30}$$

where we have used the relation

$$\sum_{k} \to [2/(2\pi)^3]\int d^3k \tag{3-31}$$

which is valid when the normalization volume tends to infinity, the factor of 2 in the numerator is due to the sum over the polarization and where

$$K(\ell,k;\omega) = \frac{1}{\pi^2(E_F + k - E_I - \omega)}[\ell^2/(\ell k)^{1/2}]\left[\frac{\beta^*(k)\beta(\ell)}{2\omega} + \frac{\pi}{m}B_f(\ell-k)\right]. \tag{3-32}$$

We now write 3-30 as

$$C(k,\omega) = \frac{e^2\beta^*(k)A}{2\pi^2k^{1/2}(E_F+k-E_I-\omega)} - e^2\int_0^{\infty}\frac{C(\ell,\omega)B_f(\ell-k)\ell^2 d\ell}{\pi m(E_F+k-E_I-\omega)(\ell k)^{1/2}} \tag{3-33}$$

where we have used 3-32 and where

$$A = -(1/\omega)\int_0^\infty C(\ell,\omega)\beta(\ell)\,\ell^{3/2}\,d\ell. \qquad (3\text{-}34)$$

Substituting 3-33 into 3-34 we obtain

$$A = \left[\frac{e^2}{\pi m}\int_0^\infty\int_0^\infty \frac{C(q,\omega)B_f(q-\ell)\beta(\ell)q^2\ell\,dq\,d\ell}{(E_F+\ell-E_I-\omega)\,(q)^{1/2}}\right]\left[\omega+\frac{e^2}{2\pi^2}\int_0^\infty\frac{|\beta(\ell)|^2\ell\,d\ell}{E_F+\ell-E_I-\omega}\right]^{-1}. \qquad (3\text{-}35)$$

We now define the reciprocal kernal, $T(\ell,k;\omega)$, for 3-33 such that [21]

$$T(\ell,k;\omega) = K(\ell,k;\omega) - e^2\int_0^\infty T(\ell,q;\omega)K(q,k;\omega)\,dq \qquad (3\text{-}36)$$

where

$$K(\ell,k;\omega) = \frac{B_f(\ell-k)\,\ell^2}{\pi m(\ell k)^{1/2}(E_F+k-E_I-\omega)}. \qquad (3\text{-}37)$$

We then obtain from 3-33 and the reciprocal kernal, $T(\ell,k;\omega)$, the following

$$C(k,\omega) = \frac{e^2\beta^*(k)A}{2\pi^2(k)^{1/2}(E_F+k-E_I-\omega)} - e^2\int_0^\infty T(k,\ell;\omega)\,\frac{e^2\beta^*(\ell)A\,d\ell}{2\pi^2(\ell)^{1/2}(E_F+\ell-E_I-\omega)} \qquad (3\text{-}38)$$

From 3-34 and 3-38 we obtain the eigenvalue equation

$$\omega = -\frac{e^2}{2\pi^2}\int_0^\infty\frac{|\beta(\ell)|^2\ell\,d\ell}{E_F+\ell-E_I-\omega} + \frac{e^4}{2\pi^2}\int_0^\infty\int_0^\infty\frac{\beta(\ell)\beta^*(q)T(\ell,q;\omega)\,\ell^{3/2}\,dq\,d\ell}{(q)^{1/2}(E_F+q-E_I-\omega)}. \qquad (3\text{-}39)$$

Although, in principle, we can solve 3-38 and 3-39 and obtain the desired results; i.e., level shift, etc.; we shall first consider a simplified version of the problem which can be solved very easily. We

can then perhaps apply the knowledge gained in solving the simpler problem to the one above.

Our simplification consists in setting $B_f(\ell-k)$ equal to zero. This is only justified by the fact that the resulting equations can be solved very easily and exactly. We then have, instead of 3-26 and 3-27 the following

$$\omega C_0(\omega) = e\sum_k [1/(2k)^{1/2}]\beta(k)C_k(\omega)$$

$$\omega C_k(\omega) = (E_F + k - E_I)C_k(\omega) + [e\beta^*(k)/(2k)^{1/2}]C_0(\omega) \qquad (3\text{-}40)$$

where now instead of 3-21 and 3-22 we have

$$E_I = E_i$$

and

$$E_F = E_f + (1/2)\sum_k k.$$

We also have, using 3-30 and 3-32,

$$C(k,\omega) = -e^2\int_0^\infty \frac{C(q,\omega)\beta^*(k)\beta(q)q^2dq}{2\pi^2(E_F+k-E_I-\omega)(kq)^{1/2}}_\omega . \qquad (3\text{-}41)$$

The eigenvalue equation 3-39 now becomes

$$\omega + (e^2/2\pi^2)\int_0^\infty [|\beta(k)|^2/(E_F+k-E_I-\omega)]kdk = 0 \qquad (3\text{-}42)$$

We now normalize $C_0(\omega)$ and $C_k(\omega)$ by requiring

$$C_0^*(\omega)C_0(\omega') + \sum_k C_k^*(\omega)C_k(\omega') = \delta_{\omega,\omega'} . \qquad (3\text{-}43)$$

From 3-40 we find

$$C_k(\omega) = \{e\beta^*(k)/[(\omega - E_F - k + E_I)(2k)^{1/2}]\}C_0(\omega). \tag{3-44}$$

Substituting 3-44 into 3-43 with $\omega = \omega'$, we obtain

$$C_0(\omega) = \{d\Delta(\omega)/d\omega\}^{-1/2} \tag{3-45}$$

where

$$\Delta(\omega) = \omega + \frac{e^2}{2\pi^2}\int_0^\infty \frac{|\beta(k)|^2 k\,dk}{E_F + k - E_I - \omega}. \tag{3-46}$$

Substituting 3-45 into 3-44 we find

$$C_k(\omega) = \{e\beta^*(k)/[(\omega - E_F - k + E_I)(2k)^{1/2}]\ [d\Delta(\omega)/d\omega]^{-1/2}. \tag{3-47}$$

From 3-23, 3-25 and 3-45 we then obtain

$$C_0(t) = \sum_\omega \exp[-i(\omega+E_I)t]\ [d\Delta(\omega)/d\omega]^{-1} \tag{3-48}$$

and

$$C_k(t) = \sum_\omega\{e\beta^*(k)/[(2k)^{1/2}(\omega-E_F-k+E_I)]\}\ [d\Delta(\omega)/d\omega]^{-1}\exp[-i(\omega+E_I)t]. \tag{3-49}$$

Letting $z = \omega+E_I-E_F$ we can then write 3-48 as[22]

$$C_0(t) = [\exp\{-iE_Ft\}/2\pi i]\oint_C \{\exp(-izt)/\Delta(z)\}dz \tag{3-50}$$

where the contour C contains all the roots of $\Delta(z)$ and $\Delta(z)$ is defined by 3-46. We must now consider $\Delta(z)$ if we wish to evaluate 3-50.

Letting

$$z = \omega + E_I - E_F \tag{3-51}$$

we can write 3-42 as

$$\Delta(z) = z - E_I + E_F + (e^2/2\pi^2)\int_0^\infty [|\beta(k)|^2/(k - z)]kdk \tag{3-52}$$

Expressions 3-50 and 3-52 are the same equations as studied by Chan[23] and we shall employ the same methods as Chan to solve them.

We can analytically continue 3-52 onto the complex plane if we can analytically continue $|\beta(k)|^2$.[24] We shall assume that $|\beta(k)|^2$ can be analytically continued and we then obtain

$$\Delta^I(z) = z - E_I + E_F + (e^2/2\pi^2)\int_0^\infty [|\beta(k)|^2/(k - z)]kdk \tag{3-53}$$

where 3-53 is defined on the first Riemann sheet for $0 < \arg(z) < 2\pi$. It can be seen that 3-53 has a cut along the positive real axis.[25] The integral in 3-50 can then be obtained by integrating along the contour, Figure 3, in the first Riemann sheet.[26]

We must find the zeros of $\Delta^I(z)$ in the lower half plane, excluding the cut. Substitution of $z = x + iy$ into 3-53 yields

$$\begin{aligned} y &= 0 \\ x &= E_I - E_F - (e^2/2\pi^2)\int_0^\infty [|\beta(k)|^2/(k - x)]kdk. \end{aligned} \tag{3-54}$$

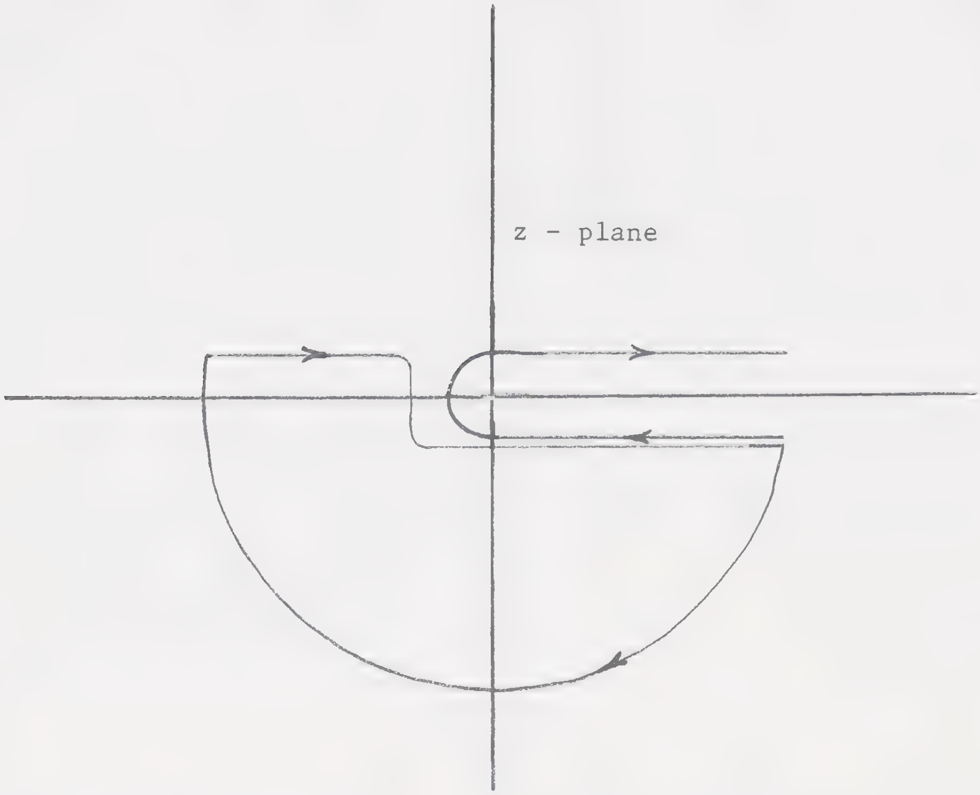

Figure 3.

Since we expect 3-50 to behave as a damped exponential which arises from a complex root of $\Delta(z)$ with a negative imaginary part we see from 3-54 that we must look onto the second Riemann sheet for the desired root.

Analytic continuation onto the second Riemann sheet, for $0 > \arg(z) > -2\pi$ is effected by defining for real $z = x$,

$$\Delta^{II}(x - i\varepsilon) = \Delta^{I}(x + i\varepsilon). \qquad (3\text{-}55)$$

Then for $z = x + iy$ we have[27]

$$\Delta^{II}(z) = \Delta^{I}(z) + (ie^2/\pi)z|\beta(z)|^2. \qquad (3\text{-}56)$$

We can now evaluate 3-50 by using the following contour.[28]

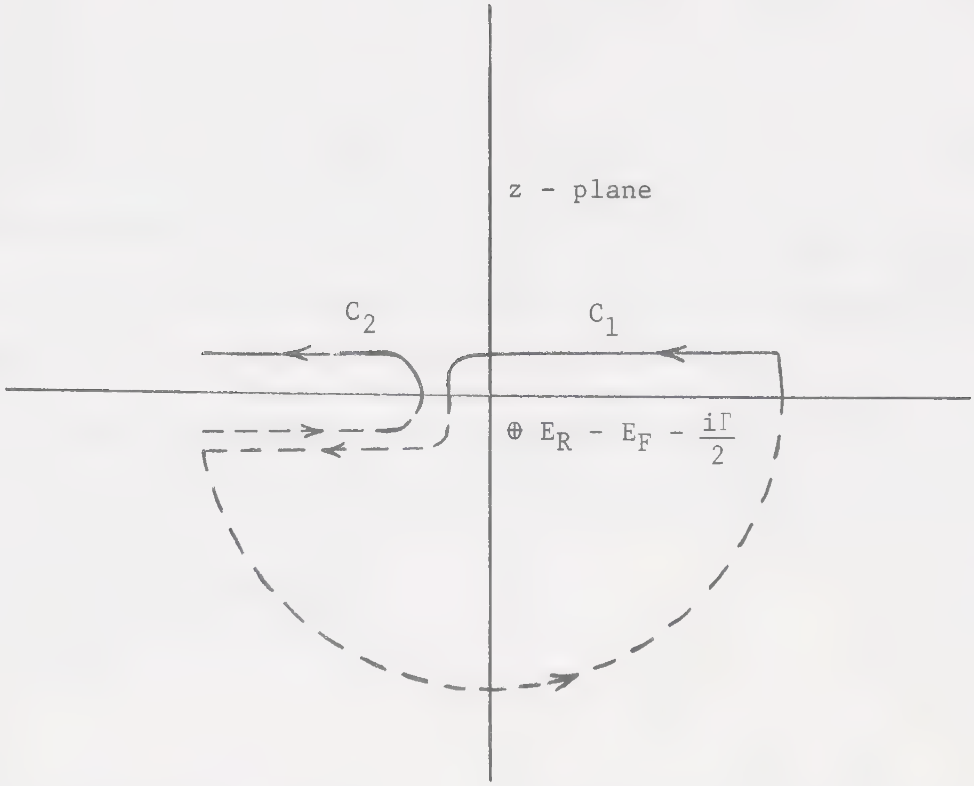

Figure 4.

The solid portion of the contour is on the first Riemann sheet while the dotted portion is on the second. We now must find the zeros of $\Delta^{II}(z)$ in the second Riemann sheet. We assume, for simplicity that $\Delta^{II}(z)$ has only one solution at

$$z = E_R - E_F - (i\Gamma/2). \tag{3-57}$$

Substitution of 3-57 into 3-56 and then equating the real and imaginary parts of 3-56 we find

$$E_R - E_I + \frac{e^2}{2\pi^2}\int_0^\infty \frac{|\beta(k)|^2 k}{(k+E_F-E_R)^2+(\Gamma^2/4)}(k + E_F - E_R)dk - \frac{e^2}{\pi}\eta = 0 \tag{3-58}$$

and

$$\frac{\Gamma}{2}\left(1 + \frac{e^2}{2\pi^2}\int_0^\infty \frac{|\beta(k)|^2 k dk}{(k+E_F-E_R)^2+(\Gamma^2/4)}\right) - \frac{e^2}{\pi}\xi = 0 \tag{3-59}$$

where

$$\xi + i\eta = [E_R - E_F - (i\Gamma/2)]|\beta[E_R - E_F - (i\Gamma/2)|^2. \qquad (3\text{-}60)$$

Equation 3-58 gives the level shift, $E_R - E_I$, and 3-59 gives the line breadth, Γ; these expressions are the same as the ones obtained by Chan and Razavy.[29]

Approximate solutions for $E_R - E_I$ and Γ, which will be useful later on, may be obtained by assuming $E_R \simeq E_I$ or $\Gamma \to 0$ we then obtain,

$$E_R - E_I \simeq -\frac{e^2}{2\pi^2} P\int_0^\infty \frac{|\beta(k)|^2 k dk}{k + E_F - E_R} \qquad (3\text{-}61)$$

and

$$\Gamma \simeq \frac{e^2}{\pi}(E_R - E_F)|\beta(E_R - E_F)|^2. \qquad (3\text{-}62)$$

The results in 3-61 and 3-62 for $E_R - E_I$ and Γ agree with that obtained by first order time-dependent perturbation calculation.[30] The symbol P stands for the principle value of the integral.

We can now write 3-50 as

$$C_0(t) = \frac{\exp(-iE_F t)}{2\pi i}\left\{\int_{C_1} + \int_{C_2}\right\}\frac{\exp(-izt)}{\Delta(z)}dz \qquad (3\text{-}63)$$

where C_1 and C_2 are shown in figure 4 and $\Delta(z)$ is either $\Delta^I(z)$ or $\Delta^{II}(z)$ depending on whether the contour is on the first or second Riemann sheet. The integrals over C_1 and C_2 are then,

$$\frac{\exp(-iE_F t)}{2\pi i}\int_{C_1}\frac{\exp(-izt)}{\Delta^{II}(z)}dz = \frac{\exp[-iE_R t-(\Gamma/2)t]}{[d\Delta^{II}(z)/dz]_{z = E_R - E_F - (i\Gamma/2)}} \qquad (3\text{-}64)$$

and

$$\frac{\exp(-iE_Ft)}{2\pi i}\int_{C_2}\frac{\exp(-izt)}{\Delta(z)}\,dz = \frac{\exp(-iE_Ft)}{2\pi i}\left\{\int_{-\infty}^{0}\frac{\exp(-ixt)}{\Delta^{II}(x)}\,dx + \int_{0}^{-\infty}\frac{\exp(-ixt)}{\Delta^{I}(x)}\,dx\right\}$$

$$= -\frac{e^2}{2\pi^2}\int_{-\infty}^{0}\frac{|\beta(x)|^2 x\exp(-ixt)\,dx}{\Delta^{I}(x)\Delta^{II}(x)}\exp(-iE_Ft)$$

(3-65)

then we can write 3-50 as

$$C_0(t) = \frac{\exp[-iE_Rt-(\Gamma/2)t]}{[d\Delta^{II}(z)/dz]_{z=E_R-E_F-(i\Gamma/2)}} - \frac{e^2}{2\pi^2}\int_{-\infty}^{0}\frac{|\beta(x)|^2 x\exp(-ixt)\,dx}{\Delta^{I}(x)\Delta^{II}(x)}\exp(-iE_Ft) \qquad (3\text{-}66)$$

where we have used 3-56 with $z = x$ (real) for the integral over C_2.

Since we have set $B_f(\ell-k)$ equal to zero we can write from 3-20

$$C_k(t) = -i\frac{e\beta^*(k)}{(2k)^{1/2}}\exp[-i(k+E_F)t]\int_0^t \exp[i(k+E_F)\tau]C_0(\tau)d\tau \qquad (3\text{-}67)$$

where we have used 3-17. We could now substitute 3-66 into 3-67 and solve for $C_k(t)$, however, the solution is so complex that no useful conclusions can be drawn from it. Therefore to obtain a useful expression for our results we write $C_0(t)$ as,

$$C_0(t) \simeq \frac{\exp[-iE_Rt-(\Gamma/2)t]}{(d\Delta^{II}(z)/dz)_{z=E_R-E_F-(i\Gamma/2)}} \qquad . \qquad (3\text{-}68)$$

From 3-56 we can write

$$\frac{d\Delta^{II}(z)}{dz} = 1 + \frac{e^2}{2\pi^2}\int_0^\infty \frac{|\beta(k)|^2 kdk}{(k-z)^2} + \frac{ie^2}{\pi}\frac{d}{dz}[z|\beta(z)|^2] \qquad (3\text{-}69)$$

thus we see that a good approximation for $\frac{d\Delta^{II}(z)}{dz}$ is,

$$\frac{d\Delta^{II}(z)}{dz} \simeq 1. \qquad (3\text{-}70)$$

Substituting 3-70 into 3-68 we then have

$$C_0(t) = \exp[-iE_R t-(\Gamma/2)t]. \qquad (3\text{-}71)$$

Substituting 3-71 into 3-67 we obtain

$$C_k(t) = -\frac{e\beta^*(k)}{(2k)^{1/2}}\left(\frac{\exp[-iE_R t-(\Gamma/2)t]-\exp[-i(k+E_F)t]}{[E_F + k - E_R + (i\Gamma/2)]}\right). \qquad (3\text{-}72)$$

Now that we have found $C_k(t)$, 3-72, we can obtain the cross-section and the oscillator strength. The oscillator strength is given by[31]

$$f_{IF}(t) = (m/2\pi^2 e^2 t)\int_{-\infty}^{\infty}|C_k(t)|^2 dk \qquad (3\text{-}73)$$

where the integral in 3-73 is over the energy variable. From 3-72 we find

$$|C_k(t)|^2 = \frac{e^2|\beta(k)|^2}{2|k|}\left\{\frac{\exp(-\Gamma t)+1-2\exp[-(\Gamma/2)t]\cos(k+E_F-E_R)t}{(k + E_F - E_R)^2 + (\Gamma/2)^2}\right\} \qquad (3\text{-}74)$$

where we obtain $|k|$, in the denominator, from $[(k)^{1/2}]^2$ since k can assume both positive and negative value. Defining $z = k + E_F - E_R$

and substituting into 3-73 and substituting the results into 3-74 we obtain

$$f_{IF}(t) = \frac{m}{4\pi^2}\frac{1}{t}\int \frac{|\beta(z-E_F+E_R)|^2}{(z-E_F+E_R)} \frac{1}{z^2+(\Gamma/2)^2}\{\exp(-\Gamma t)+1-2\exp[(-\Gamma/2)t-izt]\}dz \tag{3-75}$$

where the integral in 3-75 is done over the following contour.

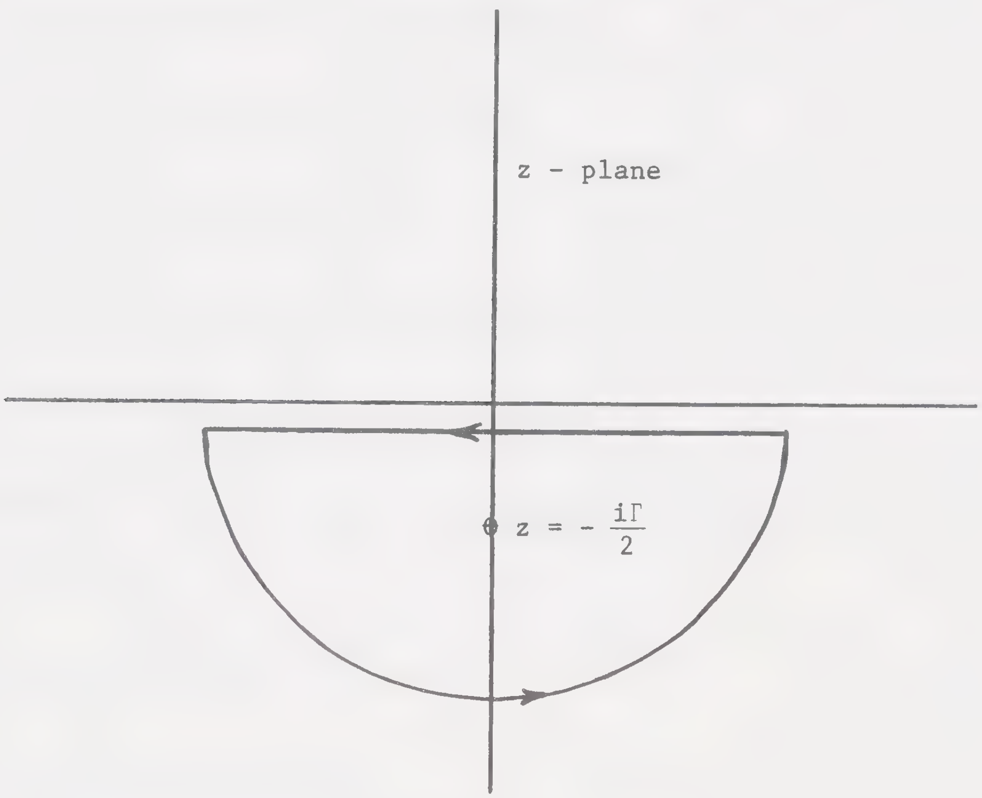

Figure 5.

The contour of figure 5 is chosen in such a way that it includes only the singularity at $z = (-i\Gamma/2)$. The selection of this particular contour is justified since in the limit $\Gamma \to 0$ the results we obtain for $\sum_I f_{IF}$, see 3-81, is the same as that obtained from perturbation theory.

We then obtain from 3-75

$$f_{IF}(t) = \frac{m}{2\pi}\frac{|\beta[E_R-E_F-(i\Gamma/2)]|^2}{[(E_R-E_F)^2+(\Gamma/2)^2]^{1/2}}\frac{1}{\Gamma t}[1-\exp(-\Gamma t)]. \tag{3-76}$$

We see that f_{IF}, 3-76 is time-dependent; this time-dependence comes from the finiteness of Γ. Taking the limit $\Gamma \to 0$ we see that 3-76 becomes time independent as is normally expected.

We know from perturbation theory that $\sum_I f_{IF} = 1$, we can obtain this result by assuming $\Gamma \to 0$, $E_R \simeq E_I$, and by considering the electric dipole transistion; note that in this approximation $E_I = E_i$ and $E_F = E_f$. We use 3-13 to calculate $\beta(k)$,

$$\beta(k) \to [i(4\pi)^{1/2}/m] \int U_I(\vec{r}) \frac{\partial}{\partial z} U_F(\vec{r}) d^3r$$

$$= -[(4\pi)^{1/2}/m] < I \mid p_z \mid F > \qquad (3\text{-}77)$$

where p_z is the mechanical momentum of the charged particle. We have[32]

$$< I \mid p_z \mid F > = im(E_I - E_F) < I \mid z \mid F >. \qquad (3\text{-}78)$$

Substituting 3-77 and 3-78 into 3-75 and taking the limit as $\Gamma \to 0$ we find the following

$$\sum_I f_{IF} = \sum_I 2m(E_I - E_F) \quad |< I \mid z \mid F >|^2 \qquad (3\text{-}79)$$

or since we have the following,[33]

$$\sum_I 2m(E_I - E_F) \quad |< I \mid z \mid F >|^2 = 1 \qquad (3\text{-}80)$$

we obtain the desired results,

$$\sum_I f_{IF} = 1 \qquad (3\text{-}81)$$

for one charged particle.

One finds that if $\overline{f_{IF}}$ denotes the time average of $f_{IF}(t)$ over the time interval $0 \leq t \leq (1/\Gamma)$ then

$$\overline{f_{IF}} = (-1) \sum_{n=1}^{\infty} \frac{(-1)^n}{n\ n!} f_{IF}(0) \simeq .7965\ f_{IF}(0) \qquad (3\text{-}82)$$

where

$$\overline{f_{IF}} = \Gamma \int_0^{(1/\Gamma)} f_{IF}(t)dt \qquad (3\text{-}83)$$

and

$$\int_0^{(1/\Gamma)} \{1-\exp(\Gamma t)\}t^{-1}dt = (-1) \sum_{n=1}^{\infty} \frac{(-1)^n}{n\ n!} \qquad (3\text{-}84)$$

and $f_{IF}(0)$ represents the oscillator strength evaluated at $t = 0$.

We can now calculate the cross-section for the emission of a photon of momentum k. The matrix element for the photon to be emitted with momentum between k and k+dk is given in the Schrödinger picture by,[34]

$$M_{IF}(k)dk = \frac{k^2dk}{\pi^2} < \phi_F(k,t = \infty) \mid H'(t = 0) \mid \phi_I(t = 0) > \qquad (3\text{-}85)$$

where

$$|\phi_F(k,t)> = C_k(t) \mid 1_k, 1_f >$$

$$|\phi_I(t)> = C_0(t) \mid 0, 1_i > \qquad (3\text{-}86)$$

and

$$H'(t=0) = e\sum_k (1/2k)^{1/2}[\beta(k)b_i^\dagger b_f a_k + \beta^*(k)a_k^\dagger b_f^\dagger b_i] \qquad (3\text{-}87)$$

Substituting 3-86 and 3-87 into 3-85 we obtain

$$M_{IF}(k)dk = \frac{ek^2dk}{\pi^2(2k)^{1/2}}\beta^*(k)C_k^*(\infty)C_0(0). \qquad (3\text{-}88)$$

Substituting 3-71 and 3-72 for $C_0(0)$ and $C_k^*(\infty)$ we find

$$|M_{IF}(k)|^2 = \frac{k^2}{4\pi^4} \frac{e^4(|\beta(k)|^2)^2}{(E_F-E_R+k)^2+(\Gamma/2)^2} \tag{3-89}$$

The flux of photons per unit frequency interval per steradian for a given polarization is $k^2/2\pi^2$,[35] thus for both polarization directions the flux is k^2/π^2. The cross-section can now be determined by the relation,

$$\sigma(k) = 2\pi \mid M_{IF}(k) \mid^2(\pi^2/k^2). \tag{3-90}$$

Substituting 3-89 into 3-90 we obtain

$$\sigma(k) = \frac{1}{2\pi} \frac{e^4(|\beta(k)|^2)^2}{(E_F-E_R+k)^2+(\Gamma/2)^2} \quad . \tag{3-91}$$

Assuming

$$E_R - E_F \simeq E_I - E_F \simeq k \tag{3-92}$$

we find, using 3-62,

$$\Gamma \simeq (e^2/\pi)k|\beta(k)|^2 \tag{3-93}$$

then substituting 3-92 and 3-93 into 3-90 we obtain

$$\sigma(k) = \frac{\pi}{2k^2} \frac{\Gamma^2}{(E_F-E_I+k)^2+(\Gamma/2)^2} \quad . \tag{3-94}$$

Eqaution 3-94 is the well-known resonance scattering cross-section.[36] The factor of 2 in the denominator arises from the two polarization states of the photon.

We now must go back and consider 3-39 and attempt to determine the level shift and line breadth as given by this equation. We shall again assume that there is a single solution of 3-39 on the second Riemann sheet and that this, as before, is located at

$$z = E_R - E_F - (i\Gamma/2) \tag{3-95}$$

or by using 3-51

$$\omega = E_R - E_I - (i\Gamma/2). \tag{3-96}$$

Analytically continuing 3-39 onto the second Riemann sheet, and substituting 3-96 into the results, we obtain

$$\begin{aligned} E_R - E_I - (i\Gamma/2) = {} & -\frac{e^2}{2\pi^2}\int_0^\infty \frac{|\beta(\ell)|^2 \ell\, d\ell}{E_F + \ell - E_F + (i\Gamma/2)} + \frac{ie^2}{\pi}\,|\beta[E_R - E_F - (i\Gamma/2)]^2[E_R - E_F - (i\Gamma/2)] \\ & + \frac{e^4}{2\pi^2}\int_0^\infty\!\!\int_0^\infty \frac{\beta(\ell)\beta^*(q)T[\ell, q; E_R - E_I - (i\Gamma/2)]\ell^{3/2}\, dq\, d\ell}{(q)^{1/2}[E_F + q - E_R + (i\Gamma/2)]} \\ & + \frac{ie^4}{\pi}\int_0^\infty \frac{\beta(\ell)\beta^*[E_R - E_F - (i\Gamma/2)]T[\ell, E_R - E_F - (i\Gamma/2); E_R - E_I - (i\Gamma/2)]\ell^{3/2}\, d\ell}{[E_R - E_F - (i\Gamma/2)]^{1/2}} \end{aligned} \tag{3-97}$$

Equating the real and imaginary parts of 3-97 we obtain

$$E_R - E_I = -\frac{e^2}{2\pi^2}\int_0^\infty \frac{|\beta(\ell)|^2 \ell}{(E_F + \ell - E_R)^2 + (\Gamma/2)^2}\,(E_F + \ell - E_R)\, d\ell + (\Gamma/2)\frac{e^2}{\pi}|\beta(E_R - E_F - (i\Gamma/2)|^2 + e^4\xi \tag{3-98}$$

$$\frac{\Gamma}{2}\left(1+\frac{e^2}{2\pi^2}\int_0^\infty \frac{|\beta(\ell)|^2\ell d\ell}{(E_F+\ell-E_R)^2+(\Gamma/2)^2}\right) = \frac{e^2}{\pi}(E_R-E_F)\left|\beta[E_R-E_F-(i\Gamma/2)]\right|^2+e^4\eta \tag{3-99}$$

where

$$\xi+i\eta = \frac{1}{2\pi^2}\int_0^\infty\int_0^\infty \frac{\beta(\ell)\beta^*(q)T[\ell,q;E_R-E_I-(i\Gamma/2)]\ell^{3/2}dqd\ell}{(q)^{1/2}[E_F+q-E_R+(i\Gamma/2)]}$$

$$+\frac{i}{\pi}\int_0^\infty \frac{\beta(\ell)\beta^*[E_R-E_F-(i\Gamma/2)]T[\ell,E_R-E_F-(i\Gamma/2);E_R-E_I-(i\Gamma/2)]\ell^{3/2}d\ell}{[E_R-E_F-(i\Gamma/2)]^{1/2}}. \tag{3-100}$$

These results for the level shift and line breadth, 3-98 and 3-99, differ from the previous results, 3-58 and 3-59, obtained with a simpler model, by a term proportional to e^4.

Expressions 3-98 and 3-99 along with 3-38 are the solution to our problem. While these results are not simple, they can be solved by numerical methods.

CHAPTER IV. DEPENDENCE OF THE LEVEL SHIFT AND LINE BREADTH ON THE BINDING POTENTIAL

In this chapter we shall determine what, if any, effect the shape of the binding potential has on the level shift and line breadth. We shall use for the level shift, ΔE, and line breadth, Γ, the expression 3-61 and 3-62.

We shall only consider the case where both the initial and final states are bound.

The dependence of the level shift and line breadth on the potential is through the dependence of $\beta(k)$, 3-13, on the wave functions $U_i(\vec{r})$ and $U_f(\vec{r})$, which in turn depend on the potential. In the first section we derive expressions for the level shift and line breadth for general forms of $U_i(\vec{r})$ and $U_f(\vec{r})$. In the second and third sections we have assumed specific form for $U_i(\vec{r})$ and $U_f(\vec{r})$ and from these determined the level shift and line breadth.

1. Calculation of $\beta(k)$ and General Expressions for ΔE and Γ.

From 3-13 we have

$$\beta(k) = (i/m)(4\pi)^{1/2}\int \exp(i\vec{k}\cdot\vec{r})U_i(\vec{r})\vec{e}(k)\cdot\nabla U_f(\vec{r})d^3r. \tag{4-1}$$

Taking ψ as our polar angle and taking $\vec{e}(k)$ in the z direction, i.e. $\vec{e}(k) = (0,0,1)$ and $\vec{k}$ in the x direction, i.e. $\vec{k} = (k,0,0)$ we then obtain

$$\beta(k) = (i/m)(4\pi)^{1/2}\int \exp(ikr\sin\theta\cos\psi)U_i(\vec{r})\cos\theta(\partial/\partial r)U_f(\vec{r})\sin\theta r^2 d\psi d\theta dr. \tag{4-2}$$

For the sake of simplicity we will consider transitions from a P state to an S state, i.e. $U_i(\vec{r})$ and $U_f(\vec{r})$ will be written as

$$U_i(\vec{r}) = (3/4\pi)^{1/2}\cos\theta N_i R_i(r)$$
$$U_f(\vec{r}) = (1/4\pi)^{1/2} N_f R_f(r) \tag{4-3}$$

where

$$N_i^2 = [\int_0^\infty |R_i(r)|^2 r^2 dr]^{-1}$$
$$N_f^2 = [\int_0^\infty |R_f(r)|^2 r^2 dr]^{-1}. \tag{4-4}$$

Substituting 4-3 into 4-2 we then obtain

$$\beta(k) = (i/m)(3/4\pi)^{1/2} N_i N_f \int \exp(ikr\sin\theta\cos\psi) R_i(r)[(\partial/\partial r)R_f(r)] \cos^2\theta\sin\theta\, r^2 d\psi d\theta dr. \tag{4-5}$$

The integral over ψ is given by[37]

$$\int_0^{2\pi} \exp(ikr\sin\theta\cos\psi)d\psi = 2\pi J_0(kr\sin\theta). \tag{4-6}$$

The integral over θ is then given by[38]

$$\int_0^{2\pi} J_0(kr\sin\theta)\cos^2\theta\sin\theta d\theta = 2\int_0^1 J_0(krx)(1-x^2)^{1/2} x dx$$
$$= (2)^{1/2}\Gamma(3/2)[2/(kr)^{3/2}]J_{3/2}(kr). \tag{4-7}$$

We now can write 4-5 as

$$\beta(k) = (i\pi/m)N_i N_f (6)^{1/2} \int_0^\infty \frac{J_{3/2}(kr)}{(kr)^{3/2}} R_i(r)[(\partial/\partial r)R_f(r)]r^2 dr. \tag{4-8}$$

This is the desired expression for $\beta(k)$.

From 3-61 and 3-62 we have

$$\Delta E = E_R - E_I = -\frac{e^2}{2\pi^2} P\int_0^\infty \frac{|\beta(k)|^2 kdk}{k+E_F-E_R} \tag{4-9}$$

$$\Gamma = (e^2/\pi)(E_R - E_F)\ |\beta(E_R - E_F)|^2 \tag{4-10}$$

where from 4-8 we have

$$|\beta(k)|^2 = \frac{6\pi^2}{m}[N_i N_f]^2 \int_0^\infty\int_0^\infty \frac{J_{3/2}(kr)J_{3/2}(kr')}{k^3(rr')^{3/2}}$$

$$[R_i(r)\frac{\partial}{\partial r}R_f(r)][R_i(r')\frac{\partial}{\partial r'}R_f(r')]r^2r'^2drdr'. \tag{4-11}$$

From 4-9, 4-10 and 4-11, with an assumed form for $R_i(r)$ and $R_f(r)$, the expressions for ΔE and Γ can now be calculated. We now turn to two specific examples of this calculation.

2. The Atomic System.

We shall assume that the normalized radial wave functions $R_i(r)$ and $R_f(r)$ are given by

$$R_i(r) = [2\mu^{5/2}/(3)^{1/2}]r\exp(-\mu r)$$

$$(4\text{-}12)$$

$$R_f(r) = 2\nu^{3/2}\exp(-\nu r)$$

where μ and ν are proportional to the energy of the particle and are given by[39]

$\mu = -(4E_i/e^2)$

$\nu = -(2E_f/e^2)$

We notice that the dimensions of μ and ν are fm^{-1} if energy is measured in Mev and e^2 has dimensions Mev fm.

Substituting 4-12 into 4-8 we obtain

$$\beta(k) = -\frac{i\pi}{m}(32)^{1/2}(\mu\nu)^{5/2}\int_0^\infty \frac{J_{3/2}(kr)}{(kr)^{3/2}} r^3 \exp[-(\mu+\nu)r]dr \qquad (4\text{-}13)$$

The integral over r in 4-13 is given by[40]

$$\int_0^\infty \frac{J_{3/2}(kr)}{(kr)^{3/2}} r^3 \exp[-(\mu+\nu)r]dr = \frac{1}{(8)^{1/2}}\frac{8}{(\pi)^{1/2}}\frac{1}{[(\mu+\nu)^2+k^2]^2}. \qquad (4\text{-}14)$$

Substituting 4-14 into 4-13 we obtain the final form of $\beta(k)$

$$\beta(k) = -\frac{16i(\pi)^{1/2}}{m}(\mu\nu)^{5/2}\frac{1}{[(\mu+\nu)^2+k^2]^2}. \qquad (4\text{-}15)$$

Substituting 4-15 into 4-9 we obtain

$$\Delta E = -\frac{128e^2}{m^2\pi}(\mu\nu)^5 \, P\int_0^\infty \frac{kdk}{[(\mu+\nu)^2+k^2]^4[k+E_F-E_R]}. \qquad (4\text{-}16)$$

Substitution of 4-15 into 4-10 gives

$$\Gamma = \frac{256e^2}{m^2}(\mu\nu)^5\frac{(E_R-E_F)}{[(\mu+\nu)^2+k^2]^4}. \qquad (4\text{-}17)$$

Putting $\hbar$ and c in 4-16 and 4-17 where they should occur we obtain

$$\Delta E = -\frac{128}{\pi}\frac{\hbar^2e^2}{m^2c^2}(\mu\nu)^5 \, P\int_0^\infty \frac{kdk}{[(\mu+\nu)^2+k^2]^4[k+(E_R-E_F/\hbar c)]} \qquad (4\text{-}18)$$

$$\Gamma = \frac{256\hbar e^2}{c^2 m^2} (\mu\nu)^5 \frac{(E_R - E_F/\hbar c)}{\{(\mu+\nu)^2 + [(E_R - E_F)^2/\hbar^2 c^2]\}^4} \qquad (4\text{-}19)$$

then ΔE is given in units of Mev and Γ in units of $\sec^{-1}$.

We shall assume that $E_R = E_i$ since the difference $E_R - E_i$ is of order e^2 and in this approximation correction of this order can be neglected. We notice that 4-18 and 4-19 are symmetric in μ and ν; therefore we shall only consider the case where the final energy, i.e. ν, is kept constant and the initial energy, i.e. μ, is allowed to vary. The fact that 4-18 and 4-19 are symmetric is not in general true, as will be seen in section 3. It is obvious, from 4-8, that $\beta(k)$ is not in general symmetric; thus 4-18 and 4-19 are not in general symmetric.

Considering 4-18 and 4-19 as $\mu \to \infty$ we find

$$\Delta E \sim (1/\mu^4)$$

$$(4\text{-}20)$$

$$\Gamma \sim (1/\mu^2)$$

For $\mu \to 0$ we find

$$\Delta E \sim \mu^5$$

$$(4\text{-}21)$$

$$\Gamma \sim \mu^6$$

Since both ΔE and Γ are finite everywhere we then obtain the following figures.

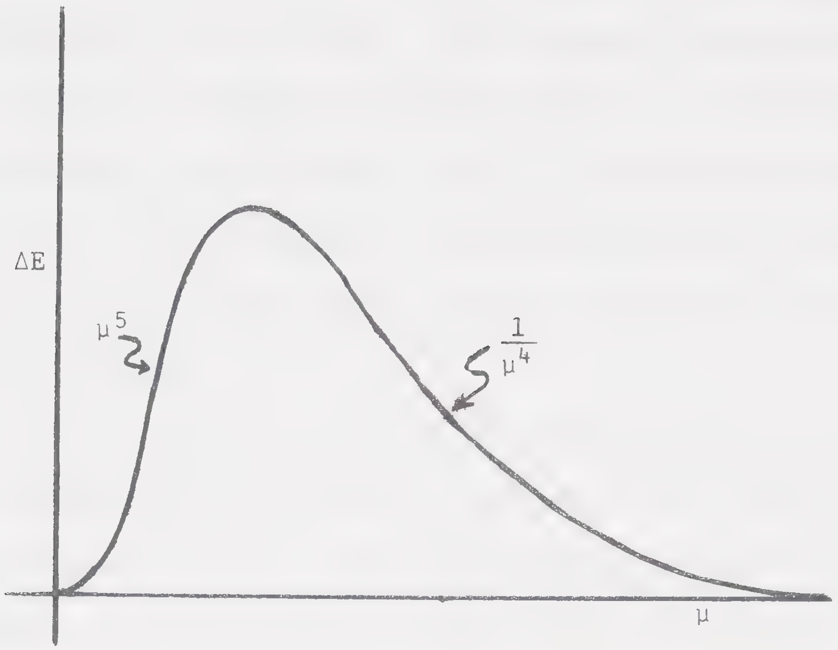

Figure 6.

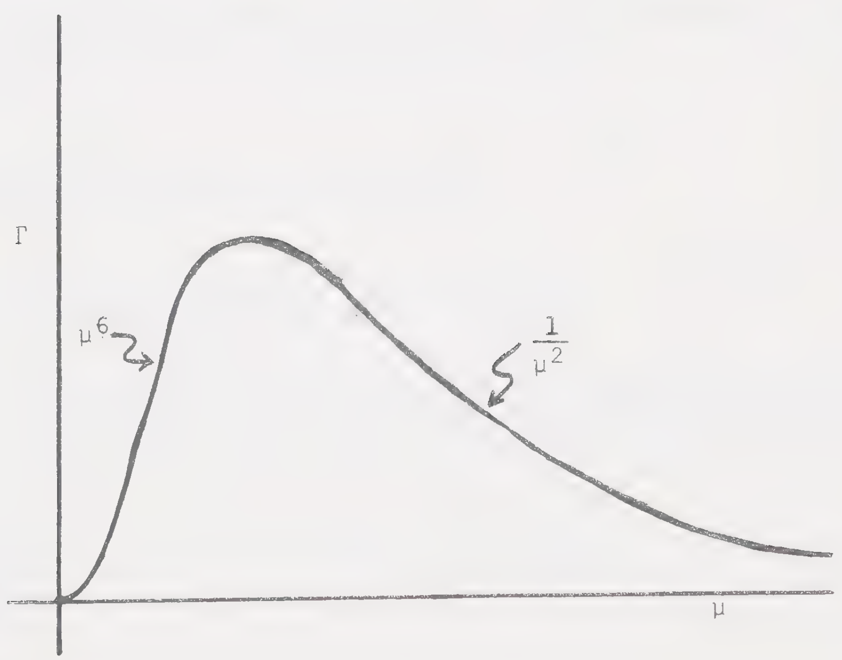

Figure 7.

Therefore, we see that if μ is large, i.e. a strongly bound state, then ΔE and Γ both are small. This is due to the fact that the particle is bound very tightly, hence the effect of the electromagnetic field on its energy is very slight. When μ is very small we again obtain the results of ΔE and Γ being small, this corresponds to a free particle; this is due to the fact that a free particle cannot absorb one photon and conserve momentum.

There is also a very pronounced region of values μ may have where we obtain quite large values for ΔE and Γ. This region corresponds to a loosely bound particle. This arises because the particle is not tightly bound and thus the electromagnetic field can exert more influence upon its energy.

3. The Nuclear System.

We take the normalized radial wave functions $R_i(r)$ and $R_f(r)$ to be[41]

$$R_i(r) = (\mu^5/\pi)^{1/4}\,(8/3)^{1/2}\,r\,\exp(-\mu r^2/2)$$

$$R_f(r) = 2(\nu^3/\pi)^{1/4}\,\exp(-\nu r^2/2) \qquad (4\text{-}22)$$

i.e. the 1S and 1P states for a harmonic oscillator potential well, where[42]

$$\mu = (2/5)mE_i$$

$$\nu = (2/3)mE_f\ .$$

We notice that μ and ν have units of fm^{-2} if energy is in Mev and mass in Mev sec fm^{-1}.

Substituting 4-22 into 4-8 we obtain

$$\beta(k) = -\frac{4i\pi^{1/2}}{m}(\nu^7\mu^5)^{1/4}\int_0^\infty \frac{J_{3/2}(kr)}{(kr)^{3/2}}\, r^4 \exp\{-(\mu+\nu)r^2/2\}\, dr. \tag{4-23}$$

The integral over r in 4-23 is given by[43]

$$\int_0^\infty \frac{J_{3/2}(kr)}{(kr)^{3/2}}\, r^4 \exp\{-(\mu+\nu)r^2/2\}\, dr = (\mu+\nu)^{-5/2}\exp\{-k^2/2(\mu+\nu). \tag{4-24}$$

Substituting 4-24 into 4-23 we obtain for $\beta(k)$

$$\beta(k) = -\frac{4i\pi^{1/2}}{m}\frac{(\nu^7\mu^5)^{1/4}}{(\mu+\nu)^{5/2}}\exp\{-k^2/2(\mu+\nu)\}\ . \tag{4-25}$$

Substituting 4-25 into 4-9 we obtain

$$\Delta E = -\frac{16e^2}{\pi m^2}(\nu^7\mu^5)^{1/2}\, P\int_0^\infty \frac{k\exp\{-k^2/(\mu+\nu)\}\, dk}{(k+E_F-E_R)\ (\mu+\nu)^5}\ . \tag{4-26}$$

Substitution of 4-25 into 4-10 gives

$$\Gamma = \frac{32e^2}{m^2}(\nu^7\mu^5)^{1/2}\ (E_R-E_F)\ \frac{\exp\{-(E_R-E_F)^2/(\mu+\nu)\}}{(\mu+\nu)^5}\ . \tag{4-27}$$

Putting $\hbar$ and c into 4-26 and 4-27 where they should occur we obtain

$$\Delta E = -\frac{16\hbar^2e^2}{\pi m^2c^2}(\nu^7\mu^5)^{1/2}\, P\int_0^\infty \frac{k\exp\{-k^2/(\mu+\nu)\}\, dk}{[k+(E_F-E_R)/\hbar c](\mu+\nu)^5} \tag{4-28}$$

$$\Gamma = \frac{32e^2\hbar}{c^2m^2}(\nu^7\mu^5)^{1/2}\ \frac{E_R-E_F}{\hbar c}\ \frac{\exp\{-(E_R-E_F)^2/\hbar^2c^2(\mu+\nu)\}}{(\mu+\nu)^5} \tag{4-29}$$

then ΔE is given in units of Mev and Γ in units of sec^{-1}.

We shall assume that $E_R = E_i$ since the difference $E_R = E_i \sim e$. Since 4-28 and 4-29 are not symmetric we have two cases to consider. First, where ν, the final energy, is fixed and μ, the initial energy, varies; second where μ is constant and ν varies.

1. μ fixed

   a. $\nu \to \infty$

      $\Gamma \sim \exp(-\nu)/\nu^{1/2}$

      $\Delta \sim 1/\nu^{5/2}$

   b. $\nu \to 0$

      $\Gamma \sim \nu^{9/2}$

      $\Delta E \sim \nu^{7/2}$

2. ν fixed

   a. $\mu \to \infty$

      $\Gamma \sim \exp(-\mu)/\mu^{3/2}$

      $\Delta E \sim 1/\mu^{7/2}$

   b. $\mu \to 0$

      $\Gamma \sim \mu^{7/2}$

      $\Delta E \sim \mu^{5/2}$

Since both ΔE and Γ are finite everywhere we then obtain the following figures.

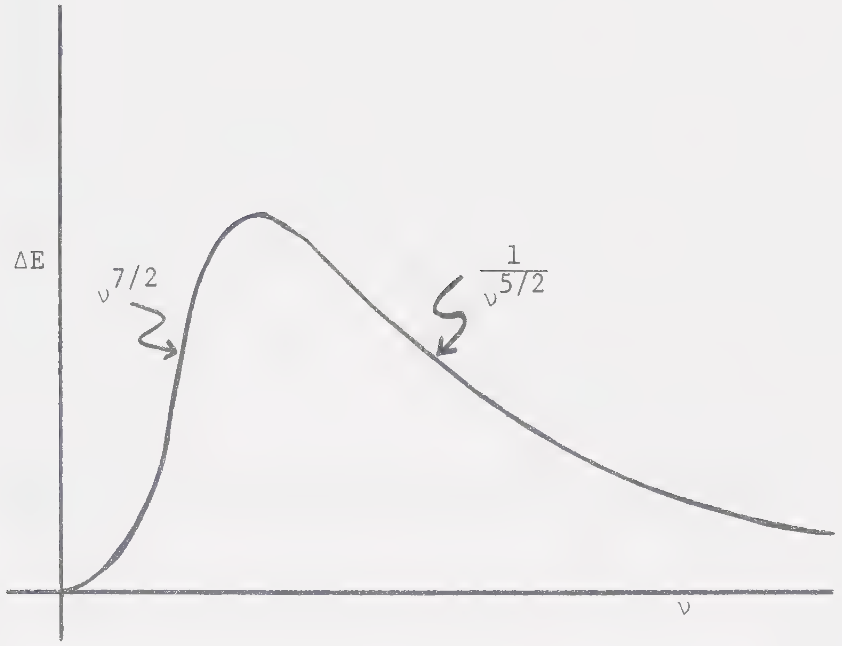

Figure 8.

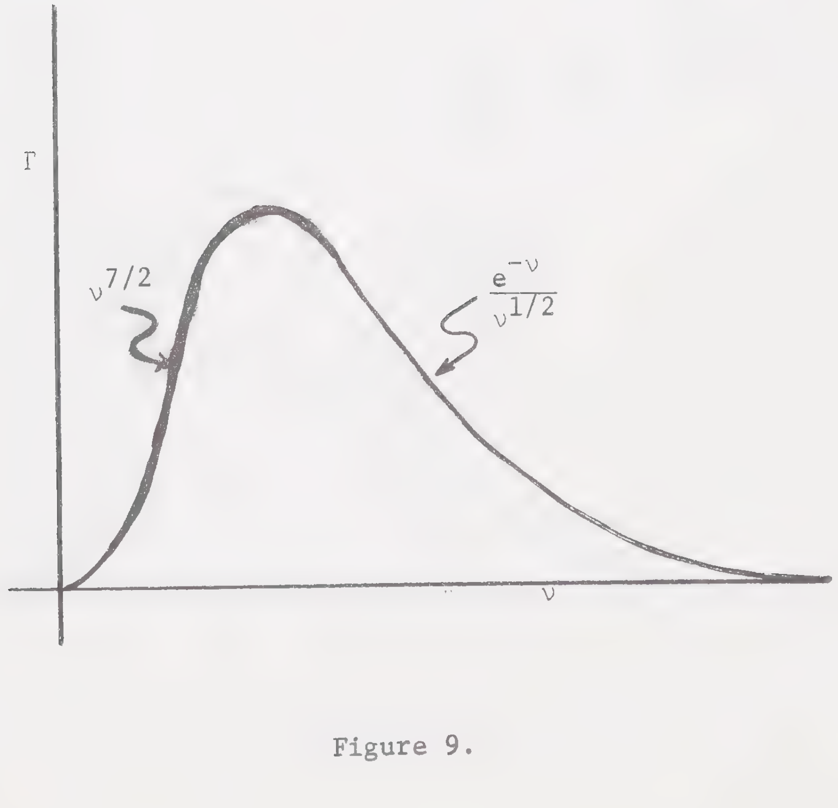

Figure 9.

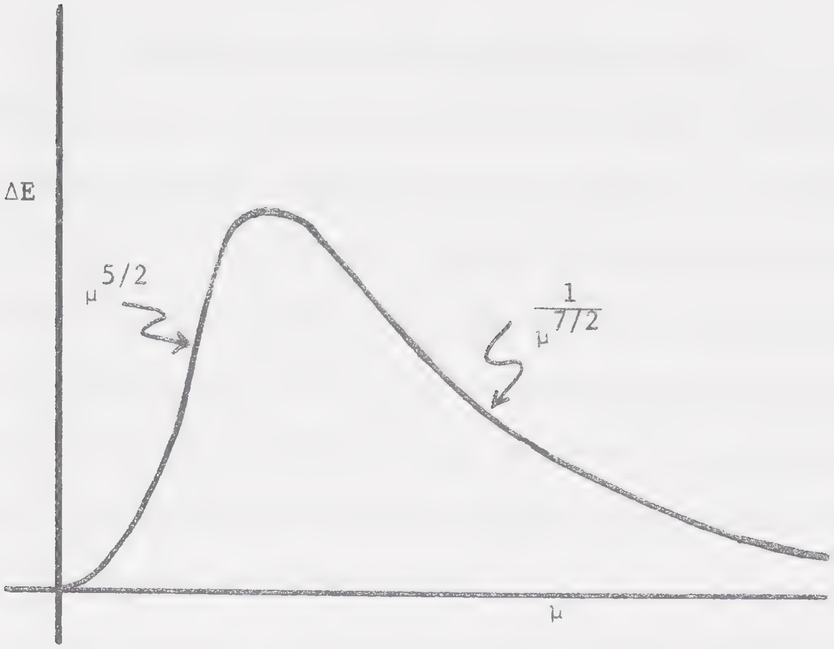

Figure 10.

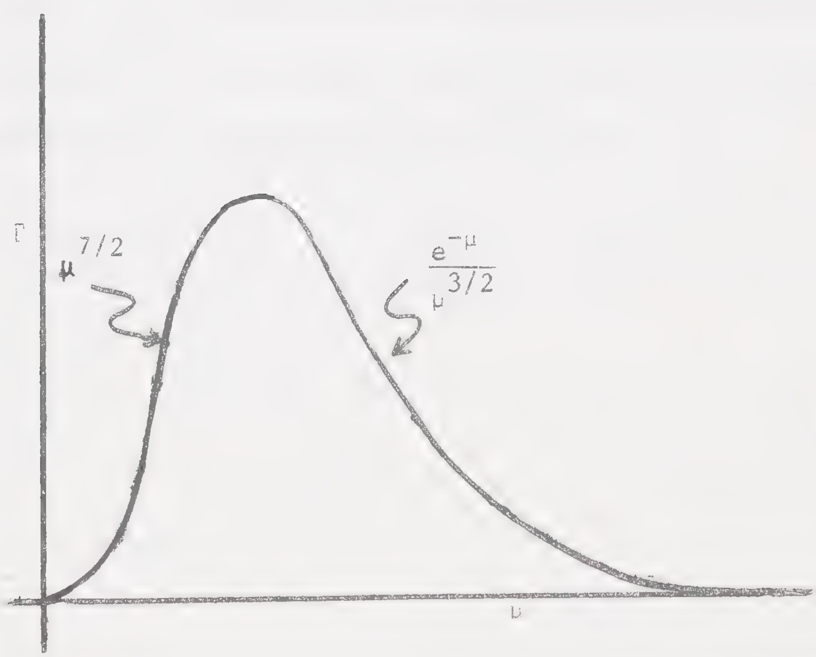

Figure 11.

From figures 8-11 we notice that the dependence of ΔE and Γ on the binding energy is of the same form as for the atomic case, section 2. Thus, if the nuclear potential is very strong, i.e. large binding energies, ΔE and Γ are quite small. This is due to the fact that the particle is bound very tightly, hence the effect of the electromagnetic field, on the particle, is very slight. We notice that if the binding energy is very small, i.e. the particle is almost free, then ΔE and Γ go to zero. This arises from the fact that a free particle cannot absorb one photon and conserve momentum.

There is also a range of binding potentials where we may have very large values of ΔE and Γ. This region corresponds to a loosely bound particle. The large values of ΔE and Γ obtained in this region arises because the particle is influenced more by the electromagnetic field.

While this analysis cannot discriminate between two similar potentials or give the exact shape of the potential it can be used to determine whether the potential is strong or weak.

CHAPTER V. SUMMARY AND CONCLUSIONS

In this thesis we have considered the problem of a bound, charged particle interacting with an electromagnetic field. We have seen, 2-48 and 3-61, that the correction, to the frequency for the classical particle and the energy level for the quantum mechanical particle, are both of order e^2.

For the quantum mechanical model we obtained not only the level shift but also the line breadth, probability of transition, and the cross-section; these quantities all agreed, in the lowest order of e^2, with previous well-known results of perturbation theory. This agreement with previous work indicates that the model used, 3-4, conforms somewhat to the actual physical world.

In chapter IV we used the results obtained in chapter III to investigate the dependence of the line breadth and level shift on the binding potential. The results obtained from this investigation were in agreement with what one would expect, i.e. for a very tightly bound particle the line breadth and level shift were very small while for more loosely bound particles we saw that the line breadth and level shift became quite large.

From the results of chapter IV we then can make some general statements about the nuclear potential. First, if the experimental values of line breadth and level shift of nuclear systems are small then the potential which binds the nucleons in the nucleus is very strong, somewhat like a very deep square well. And vice versa if the line breadth and level shift are large then the nuclear potential

would resemble a shallow square well.

It is not clear, as yet, how much more information about the nuclear potential one can obtain by the type of analysis as done in chapter IV. However, since the atomic system is quite well understood this might serve as a guide for further and deeper studies into the connection between level shift, line breadth, and the binding potential

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